INTEGRATING RATIONAL FUNCTIONS

To find $\int \frac{N(x)}{D(x)}dx$, where $N(x)$ and $D(x)$ are both polynomials, and no cancellation is possible

Some rational integrands require partial fractions, others do not, and still others are a combination of the two.
The process below outlines when you should and should not use partial fractions.

If degree of $N(x) \geq$ degree of $D(x)$ (ie. numerator has same degree as, or higher degree than, denominator)

1. Perform polynomial long division
2. Rewrite integrand as polynomial + remainder (with degree < degree of $D(x)$)
3. Use processes below to find the integral of $\frac{remainder}{D(x)}$ (ie. set $N(x) = remainder$ and continue)

If degree of $D(x) = 1$ (ie. denominator is linear)

1. Let $u = D(x)$ & perform $u-$substitution
   (or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

eg. $\int \frac{13}{7x-5}dx$ Let $u = 7x-5$, so $\frac{du}{dx} = 7$ and $dx = \frac{1}{7}du$

$= \int \frac{13}{7}\frac{du}{u} = \frac{13}{7}\ln|u| + C = \frac{13}{7}\ln|7x-5| + C$

If $N(x) = k \cdot D'(x)$ (ie. numerator is constant multiple of derivative of denominator) **TYPE 1**

1. Let $u = D(x)$ & perform $u-$substitution
   (or use guess & check – antiderivative is a multiple of $\ln|D(x)|$)

eg. $\int \frac{20-5x}{x^2-8x+25}dx$ Let $u = x^2-8x+25$, so $\frac{du}{dx} = 2x-8$ and $20-5x = -\frac{5}{2}(2x-8)$

$= \int -\frac{5}{2}\frac{du}{u} = -\frac{5}{2}\ln|u| + C = -\frac{5}{2}\ln|x^2-8x+25| + C$

If degree of $D(x) = 2$ and is irreducible
(ie. denominator is quadratic with negative discriminant, so denominator has no real roots / only complex roots)

If $N(x) = c$ (ie. numerator is constant) **TYPE 2**

1. Factor leading coefficient from $D(x)$ (ie. so denominator starts with x^2)
2. Complete the square for $D(x) = (x+h)^2 + a^2$
3. Factor a^2 from denominator, let $u = \frac{x+h}{a}$ & perform $u-$substitution

   (or use $\int \frac{1}{(x+h)^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x+h}{a}$)

eg. $\int \frac{7}{2x^2-16x+50}dx = \frac{7}{2}\int \frac{1}{x^2-8x+25}dx = \frac{7}{2}\int \frac{1}{(x-4)^2+3^2}dx = \frac{7}{6}\tan^{-1}\frac{x-4}{3} + C$

If $N(x) = ax + b$ (ie. numerator is linear) **TYPE 3**

1. Use technique similar to partial fractions shortcut to rewrite numerator as $A \cdot D'(x) + B$
   (ie. constant multiple of derivative of denominator + constant)
   (see Partial Fractions Decomposition handout (special note regarding Math 1B))
2. Split integrand into integrand of **TYPE 1** + integrand of **TYPE 2**
3. Use processes above (NOTE: no absolute values required in $\ln(D(x))$)

eg. $\displaystyle\int \frac{3x-17}{x^2-8x+25}\,dx = \int \frac{A(2x-8)+B}{x^2-8x+25}\,dx$ $\quad x=4:\quad 3(4)-17 = A(0)+B \Rightarrow B=-5$
$\quad$ *coefficient of* x: $\quad 3 = 2A \Rightarrow A = \frac{3}{2}$

$$= \int \frac{\frac{3}{2}(2x-8)-5}{x^2-8x+25}\,dx = \frac{3}{2}\int \frac{2x-8}{x^2-8x+25}\,dx - 5\int \frac{1}{(x-4)^2+3^2}\,dx$$

$$= \frac{3}{2}\ln(x^2-8x+25) - \frac{5}{3}\tan^{-1}\frac{x-4}{3} + C$$

NOTE: TYPE 3 rational functions can also be integrated using a trigonometric substitution.

eg. $\displaystyle\int \frac{3x-17}{x^2-8x+25}\,dx = \int \frac{3x-17}{(x-4)^2+9}\,dx$ $\quad$ Let $x = 4+3\tan\theta$, so $dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{3(4+3\tan\theta)-17}{(3\tan\theta)^2+9} 3\sec^2\theta\, d\theta = \int \frac{9\tan\theta-5}{3}\,d\theta = 3\ln|\sec\theta| - \frac{5}{3}\theta + C$$

$$= 3\ln\sqrt{(x-4)^2+9} - \frac{5}{3}\tan^{-1}x + C = \frac{3}{2}\ln((x-4)^2+9) - \frac{5}{3}\tan^{-1}x + C$$

All other cases require partial fractions

1. Perform partial fractions decomposition (see handout)
   NOTE: for irreducible quadratic denominators $d(x) = ax^2+bx+c$
   (or powers of these factors, ie. $[d(x)]^n$ or $(ax^2+bx+c)^n$)
   write numerator in $A \cdot d'(x) + B$ form for **TYPE 3** to save work later on
2. For all partial fractions with linear and irreducible quadratic denominators:
   Use processes above
3. For all partial fractions with denominator $[d(x)]^n = (ax+b)^n$ (ie. power of linear factor):
   Let $u = ax+b$ & perform $u-$substitution
   (or use guess & check – antiderivative is a multiple of $\dfrac{1}{(ax+b)^{n-1}}$)
4. For all partial fractions with denominator $[d(x)]^n = (ax^2+bx+c)^n$ (ie. power of irreducible quadratic factor):
   Split integrand into integrand with numerator $A \cdot d'(x)$ + integrand with numerator B
   For first integrand:
   Let $u = ax^2+bx+c$ & perform $u-$substitution
   (or use guess & check – antiderivative is a multiple of $\dfrac{1}{(ax^2+bx+c)^{n-1}}$)
   For second integrand:
   Factor leading coefficient from ax^2+bx+c (ie. so irreducible quadratic starts with x^2)
   Complete the square for $x^2+Bx+C = (x+h)^2+k^2$
   Let $x+h = k\tan\theta$ & perform trigonometric substitution
   NOTE: This is the hardest type – there will be no required problems of this type on tests

Practice against the following examples:

TYPE 1

$$\int \frac{7}{3x+8}\,dx = \frac{7}{3}\ln|3x+8| + C$$

TYPE 1

$$\int \frac{6-9x}{3x^2-4x-4}\,dx = -\frac{3}{2}\ln|3x^2-4x-4| + C$$

TYPE 2

$$\int \frac{5}{4x^2+24x+52}\,dx = \frac{5}{4}\int \frac{1}{x^2+6x+13}\,dx = \frac{5}{4}\int \frac{1}{(x+3)^2+2^2}\,dx = \frac{5}{8}\tan^{-1}\frac{x+3}{2} + C$$

TYPE 3

$$\int \frac{5x-7}{x^2+8x+25}\,dx = \int \frac{\frac{5}{2}(2x+8)-27}{x^2+8x+25}\,dx = \frac{5}{2}\int \frac{2x+8}{x^2+8x+25}\,dx - 27\int \frac{1}{(x+4)^2+3^2}\,dx$$

$$= \frac{5}{2}\ln(x^2+8x+25) - 9\tan^{-1}\frac{x+4}{3} + C$$

MIXED

$$\int \frac{-20x-12}{(x+1)^2(x^2+4x+7)}\,dx = \int\left(\frac{-6}{x+1} + \frac{2}{(x+1)^2} + \frac{3(2x+4)+4}{x^2+4x+7}\right)dx$$

$$= \int \frac{-6}{x+1}\,dx + \int \frac{2}{(x+1)^2}\,dx + 3\int \frac{2x+4}{x^2+4x+7}\,dx + \int \frac{4}{(x+2)^2+(\sqrt{3})^2}\,dx$$

$$= -6\ln|x+1| - \frac{2}{x+1} + 3\ln(x^2+4x+7) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}} + C$$

MIXED

$$\int \frac{3x^3-11x^2-49x}{x^2-5x-6}\,dx = \int\left(3x+4+\frac{-11x+24}{(x-6)(x+1)}\right)dx = \int\left(3x+4-\frac{6}{x-6}-\frac{5}{x+1}\right)dx$$

$$= \frac{3}{2}x^2 + 4x - 6\ln|x-6| - 5\ln|x+1| + C$$